International Journal of Theoretical Physics, Vol. 29, No. 6, 1990

Cosmological Models in General Relativity and Brans–Dicke Theories: A Comparison

Marcelo Samuel Berman[1,2]

Received December 27, 1989

We review general relativistic and Brans–Dicke cosmological models of the early universe and for the present phase. Both theories render similar results, in general, as far as Mach's principle is concerned. There is some difference in the stability problem for the inflationary phase, and we point out how to test one theory against the other experimentally.

1. INTRODUCTION

We shall review cosmological solutions in general relativistic (GR) and Brans–Dicke (BD) theories, comparing results and suggesting experimental tests for deciding between them. Let us begin with Mach's principle.

Though Brans and Dicke (1961) devised their alternative theory in order that it give Machian solutions, I shall show below that as far as cosmology is concerned, both general relativistic and BD models yield Machian solutions. It is worth noticing that Brans and Dicke, in their original paper, only proved the Machian property for a very particular cosmological solution. What is meant by a cosmological solution obeying Mach's principle?

The Whitrow–Randalls relation (Whitrow, 1946; Whitrow and Randall, 1951; Berman and Som, to appear), attributed by Brans and Dicke to Sciama, is given by

$$\frac{GM_{\mathrm{H}}}{R_{\mathrm{H}}} \sim 1 \tag{1}$$

[1]Department of Astronomy, University of Florida, Gainesville, Florida 32611.
[2]Permanent address: Departamento de Ciencias Exatas da Faculdade de Filosofia, Ciencas e Letras da FURJ, Joinville, SC 89200, Brazil.

0020-7748/90/0600-0571$06.00/0

where G is the Newtonian gravitational "constant," and M_H stands for the observable mass at the Hubble length R_H.

For a Euclidean universe,

$$\rho = \frac{M_H}{\frac{4}{3}\pi R_H^3} \tag{2}$$

where ρ stands for rest energy density; on the other hand, the Hubble length is defined by

$$R_H = H^{-1} \tag{3}$$

where H is Hubble's parameter.

From (1)-(3) we obtain

$$G\rho \sim H^2 \tag{4}$$

This will be taken as the condition for a Machian cosmological solution.

2. THE PHASES OF THE UNIVERSE

Grøn (1986) has remarked that it is current belief that the universe had the following early phases:

$$R_1 \propto t^{1/2}, \qquad p = \rho/3 \tag{5}$$

$$R_2 \propto e^{Ht} \quad \text{(inflation)} \qquad p = -\rho \tag{6}$$

$$R_3 \propto t^{1/2} \qquad p = \rho/3 \tag{7}$$

On the other hand, according to Schwarzschild (1987), the present phase of the universe is

$$R_4 \propto t^{2/3}, \qquad p = 0 \tag{8}$$

Berman and Gomide (1988) remarked that the above four phases are particular cases of the constant deceleration parameter type. Indeed, if we call the deceleration parameter q, and suppose it to be constant, we have, on integration, that

$$R(t) = (mDt)^{1/m} \qquad \text{with } m \neq 0 \tag{9}$$

and

$$R(t) = R_0 e^{Ht} \qquad \text{with } m = 0 \tag{10}$$

where

$$m = q + 1 \tag{11}$$

and D, R_0 are constants.

In the $m \neq 0$ case, we also have

$$H = 1/mt \tag{12}$$

It is evident that $m = 0$ stands for inflation, while $m = 2$ and $m = 3/2$ stand, respectively, for the first (and third) and the fourth phases.

The question to be posed is whether we can find solutions with $m = 0$ and $m \neq 0$ that obey relation (4) and are thus Machian. We shall see that this is the case both within general relativity and for the BD case.

3. SOLUTIONS WITH GENERAL RELATIVITY

It is well known (Berman, 1983) that in GR we have a $k = 0$ solution for $m \neq 0$ as follows:

$$\rho = \frac{3}{8\pi G m^2} t^{-2} \tag{13}$$

$$p = \frac{2m-3}{3} \rho \qquad \text{(perfect gas law)} \tag{14}$$

where p stands for pressure.

Taking into account relation (12), we find that this solution is Machian, i.e., obeys relation (4).

On the other hand, for $m = 0$, we have, in general relativity,

$$\rho = \frac{3H^2}{8\pi G} \tag{15}$$

$$p = -\rho \tag{16}$$

and so this also is a Machian solution. We find, thus, that all phases in GR have Machian solutions.

4. SOLUTIONS WITH BRANS–DICKE THEORY

Let us first consider the $m = 0$ case. Berman and Som (1989) showed that, in BD theory, inflation is ruled by the following laws:

$$\rho = \rho_0 \, e^{-3H(1+\alpha)t} \tag{17}$$

$$\phi = \phi_0 \, e^{-3H(1+\alpha)t} \tag{18}$$

where ρ_0, ϕ_0, and H are constants, ϕ being the scalar field, related to G by

$$G = a^{-1}\phi^{-1} \tag{19}$$

$$a^{-1} = \frac{2w+4}{2w+3} \tag{20}$$

where w is the coupling constant and α is the constant of the equation of state

$$p = \alpha\rho \tag{21}$$

From (17)-(20) we see that

$$G\rho = \text{const} \tag{22}$$

Of course, we can arrange the constants ϕ_0 and C so as to have

$$G\rho \sim H^2$$

because H^2 is also a constant. Now let us consider the $m \neq 0$ cases. It was shown by Berman and Som (to appear) that we can have a solution of the constant deceleration type with $m \neq 0$ and

$$\rho = Ct^{-1} \tag{23}$$

$$p = (m/3 - 1)\rho \tag{24}$$

$$\phi = At \tag{25}$$

where C and A are constants to be determined and $m \neq 4$.

This solution, which solves for the power law relations in R, is evidently also Machian, after some arrangements in the arbitrary constants is made, because we have

$$G\rho = \frac{aCm^2H^2}{A}$$

So we also have in BD theory Machian solutions, as expected, if

$$\frac{aCm^2}{A} \sim 1$$

5. DENSITY PERTURBATIONS

From the above result, it is evident that both GR and the BD theory yield Machian cosmological solutions. What this tells us is that Mach's principle alone cannot be invoked in favor of the BD theory. Indeed, Brans and Dicke did not devise their framework as a new theory of gravitation, but more likely as a foil for testing GR (Adler *et al.*, 1975, Preface; M. Bazin, personal communication).

There is, however, one important difference between GR and BD solutions: while the $m \neq 0$ solutions are unstable (Berman, to appear; Berman, submitted), though not exponentially, in both theoretical frameworks, the $m = 0$ solution for GR is stable (Barrow, 1983), while for the BD case it is exponentially unstable, as was shown by Berman (1989) recently. Though in standard classical cosmology there is no acceptable theory for explaining the phase transitions, it is my opinion that growing instabilities might be associated with phase transitions, so that the BD theory has this kind of superiority over GR, not to mention the possibility of explaining galaxy formation and the present isotropy of the universe by the exponentially increasing instabilities of the $m = 0$ phase in the BD theory.

6. TESTING THE BRANS–DICKE SOLUTION

It must be observed that for the $m \neq 0$ phases, we have a perfect gas equation of state solution in both theories, with real values for the speed of sound, while for the $m = 0$ (inflation) case, only in the BD case do we have such a law, while in GR we find an annoying negative pressure. The time variation laws for p and ρ are also very different between the theories.

For the present universe, we suppose usually that $p = 0$. This leads to relation (8) in GR with $m = 3/2$, while in the BD case we would need $m = 3$, so that

$$R \propto t^{1/3} \tag{26}$$

The age of the universe, with $H_0^{-1} \approx 1.8 \times 10^{10}$ years, would be

$$t_{\rm GR} \cong 1.2 \times 10^{10} \text{ years}$$

$$t_{\rm BD} \cong 6 \times 10^{9} \text{ years}$$

On the other hand, we would have

$$\frac{\dot{G}}{G} \cong -t_0^{-1} = -3H_0^{-1} \tag{27}$$

due to (25).

Will (1987) has surveyed the experimental tests for $\dot{G}/G$, and relation (27) seems to be coherent with some of the tests. For

$$\sigma = \frac{\dot{G}}{G} \times (2 \times 10^{10} \text{ years})$$

he lists $\sigma = -(2.5 \pm 0.7)$ according to Newton, from linear occultations and eclipses; and $\sigma < 8$ according to Shapiro *et al.*, from planetary and spacecraft radar. Other results, however, give smaller values for σ than those we mention here. The most prudent attitude, it seems, is to wait for more experimental data.

A second test, in principle, could be a measure of $\dot{\rho}/\rho$, and then we would be able to decide between relations (13) and (23). For the radiation phase, $p = \rho/3$, we have a problem, because $m = 4$ is ruled out from our solution above for the BD case.

A convenient solution can be achieved by

$$m = 1$$
$$\phi = At^{-2}$$
$$\rho = Ct^{-4}$$

where A, C are constants. It can be easily checked that this is indeed a solution of the Brans-Dicke equations for the radiation phase with constant deceleration parameter in a Robertson-Walker metric with a perfect fluid and a flat universe. However, we need a negative value for the coupling constant of the theory, w. The advantages of this solution is that the horizon problem in the very early universe is solved ($\dot{R}_1 = \text{const}$). On the other hand, if we take for granted a Brans-Dicke model for the present universe, there is no need for explaining any "critical density" as in GR.

7. CONCLUSIONS

We are not able at present to choose one or the other theory as far as cosmology is concerned. More experiments should be done, as we suggest above.

ACKNOWLEDGMENTS

I am grateful to Profs. M. M. Som and Jim Ipser for reading the manuscript of this paper. This work was supported by a scholarship from the CNPq (Brazil).

REFERENCES

Adler, R., Bazin, M., and Schiffer, M. (1975). *Introduction to General Relativity*, 2nd ed., McGraw-Hill, New York.

Barrow, J. D. (1983). Perturbations of a de Sitter universe, in *The Very Early Universe*, G. W. Gibbons, S. W. Hawking, and S. T. C. Siklos, eds., Cambridge University Press, pp. 267–271.

Berman, M. S. (1983). *Nuovo Cimento*, **74B**, 182.

Berman, M. S. (1989). *Physics Letters*, **139A**, 119.

Berman, M. S. (to appear). Density perturbations in a flat universe, *General Relativity and Gravitation*.

Berman, M. S. (submitted). Density perturbations in a Brans-Dicke cosmological model.

Berman, M. S., and Gomide, F. M. (1988). *General Relativity and Gravitation*, **20**, 191.

Berman, M. S., and Som, M. M. (1989). *Physics Letters*, **136A**, 428.

Berman, M. S., and Som, M. M. (to appear). Whitrow-Randall relation and Brans-Dicke cosmology, *General Relativity and Gravitation*.

Brans, C., and Dicke, R. H. (1961). *Physical Review*, **124**, 203.

Grøn, Ø. (1986). *American Journal of Physics*, **54**, 46.

Schwarzschild, B. (1987). *Physics Today*, **40**, 17.

Whitrow, G. (1946). *Nature*, **158**, 165.

Whitrow, G., and Randall, D. (1951). *Monthly Notices of the Royal Astronomical Society*, **111**, 455.

Will, C. M. (1987). Experimental gravitation from Newton's *Principia* to Einstein's general relativity, in *300 Years of Gravitation*, S. W. Hawking and W. Israel, eds., Cambridge University Press, Cambridge, p. 100.